Low Tangential Thrust Trajectories—Improved First-Order Solution

CHONG-HUNG ZEE*
Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

Introduction

In view of recent interest in low tangential thrust spiral trajectories[1,2] and various discussions[1–4] on Zee's previous works[5,6] related to this problem, an extension of the investigations in Refs. 5 and 6 seems to be in order.

Basically, the results presented in Refs. 5 and 6 are known as the first-order approximate solution because both the stroboscopic method[5] and the method of average[6] give the behaviors of a parameter over a period of 2π of θ.† The results thus obtained depict the secular aspect of the problem. In order to have a complete first-order solution, the behavior of the parameter within the period of 2π of θ should be investigated, and its oscillatory nature can be exposed.

Constant Tangential Thrust Case (Ref. 5)

According to the asymptotic method,[7] the improved first-order approximate solution for the parameter p can be expressed as†

$$p = \bar{p} + \int [dp/d\theta - \overline{(dp/d\theta)}]d\theta$$
$$= \bar{p} + \delta p \tag{1}$$

where $\bar{p}$ is the first-order approximate solution of p, $dp/d\theta$ is shown in Eq. (R13) and $\overline{(dp/d\theta)}$ is the average value over a period of 2π of θ, i.e., Eq. (R16) with deletion of 2π on the right-hand side. The integration is carried out by assuming S, p, ψ and θ_n remain constant. As the term $\int [dp/d\theta - \overline{(dp/d\theta)}]d\theta$ is basically oscillatory in nature, its average value over a period of 2π of θ should be zero; this gives the criterion to determine the integration constant for the integral. Accordingly, Eq. (R22) gives

$$1/\bar{p} = (\bar{c}\omega/g_0) \ln[(A - \bar{\tau})/(A - \bar{\tau}_0)] + 1/\bar{p}_0 \tag{2}$$

where $\bar{p}_0$ and $\bar{\tau}_0$ are integration constants to be determined. With the aid of Eq. (R11), the integral in Eq. (1) has the form

$$\delta p = [-3\bar{S}\bar{p}^7/(A - \bar{\tau})] \sin(\theta + \bar{\psi}) \tag{3}$$

where the "bar" on various parameters designates their first-order approximations. Combining Eqs. (2) and (3) gives the improved first-order approximate solution of p; the former shows the secular portion, and the latter, the oscillatory nature.

Similarly, S has its improved first-order approximation as

$$S = \bar{S} + \delta S$$
$$= \bar{S} + [2\bar{p}^2/(A - \bar{\tau})][\sin(\theta + \bar{\psi}) - \tfrac{3}{4}\bar{S}\bar{p}^2 \sin 2(\theta + \bar{\psi})] \tag{4}$$

where Eqs. (R13) and (R24) have been employed for $dS/d\theta$, and Eq. (R26) with deletion of 2π for $\overline{(dS/d\theta)}$. Equation (R31) is written here as

$$K_1\bar{S} = 1/\bar{p}^3 \tag{5}$$

where K_1 is an integration constant to be determined. The improved first-order approximation for ψ is

$$\psi = \bar{\psi} + \delta\psi$$
$$= \bar{\psi} + [2\bar{p}^2/\bar{S}\,(A - \bar{\tau})][\cos(\theta + \bar{\psi}) - \tfrac{3}{4}\bar{S}\bar{p}^2 \cos 2(\theta + \bar{\psi})] \tag{6}$$

where Eqs. (R13) and (R25) are used for $d\psi/d\theta$, and Eq. (R27) for $\overline{(d\psi/d\theta)}$. Equation (R32) is rewritten here as

$$\bar{\psi} = \Phi_1 \tag{7}$$

and Φ_1 is an integration constant.

With the expressions of $\bar{p}$, $\bar{S}$, and $\bar{\psi}$, the first-order approximation of u is

$$\bar{u} = 1/\bar{p}^2 + \bar{S}\cos(\theta + \bar{\psi}) \tag{8}$$

and the improved first-order approximation for u can be obtained by applying a variation to u along $\bar{u}$:

$$u = 1/p^2 + S\cos(\theta + \psi)$$
$$= \bar{u} + \delta u$$
$$= \bar{u} - (2/\bar{p}^3)\delta p + \delta S \cos(\theta + \bar{\psi}) - \bar{S}\sin(\theta + \bar{\psi})\delta\psi$$
$$= 1/\bar{p}^2 + (1/K_1\bar{p}^3)\cos(\theta + \Phi_1) + [9\bar{p}/2K_1(A - \bar{\tau})]\sin(\theta + \Phi_1) \tag{9}$$

where Eqs. (3–8) have been employed. If $\bar{u}$ is approximated by $1/\bar{p}^2$, the first-order approximation between θ and τ can be obtained by combining Eqs. (R3, R6, and 2) and integrating the resulting expression‡:

$$\theta = (a/\bar{p}_0^3)(\bar{\tau} - \bar{\tau}_0) - (A - \bar{\tau}_0)a^2\{(3/\bar{p}_0^2)[X(\ln X - 1) + 1] + (3/\bar{p}_0)a[X(\ln X - 1)^2 + (X - 2)] + a^2[X(\ln X - 1)^3 + 3X(\ln X - 1) - (2X - 6)]\} \tag{10}$$

where the boundary condition $\theta = 0$ at $\bar{\tau} = \bar{\tau}_0$ is employed, and

$$a \equiv \omega\bar{c}/g_0 \qquad \text{and} \qquad X \equiv (A - \bar{\tau})/(A - \bar{\tau}_0)$$

The employment of the corresponding relationship of Eq. (1) for t (or τ) requires some manipulations. From Eq. (R3),

$$dt/d\theta = 1/u^2\omega p$$
$$\simeq (1/\bar{u}^2\omega\bar{p})(1 - 2\delta u/\bar{u})(1 - \delta p/\bar{p})$$
$$\simeq (1/\bar{u}^2\omega\bar{p})[1 - (2\delta u/\bar{u}) - (\delta p/\bar{p})] \tag{11}$$

and $\overline{(dt/d\theta)} = 1/\bar{u}^2\omega\bar{p}$. Hence

$$\tau = \bar{\tau} - (g_0/\bar{c})\int[(2\delta u/\bar{u}) + (\delta p/\bar{p})]d\theta = \bar{\tau} + [6\bar{p}^6/aK_1(A - \bar{\tau})]\cos(\theta + \Phi_1) \tag{12}$$

where $\bar{u}$ in the integrand is again approximated by $1/\bar{p}^2$, and Eqs. (3–8) are employed.

The development of the improved first-order approximate solution for the problem of a satellite subjected to a low constant tangential thrust and initially being in an orbit of small eccentricity is completed, and there are four integration constants, $\bar{p}_0$, K_1, Φ_1, and $\bar{\tau}_0$, to be determined. These four constants can be determined from the given initial conditions t (or τ) $= 0$, $\theta = 0$, $u = 1$, $du/d\theta = (du/d\theta)_0$, and $p = 1$. For the case of a satellite commencing from a circular orbit, these constants are

$$\bar{p}_0 = 1 - (6/A^2) \qquad K_1 = -(A/2)[1 - (9/2A^2)]$$
$$\Phi_1 = (\pi/2) - (3/2A) \qquad \bar{\tau}_0 = 18/A^3a \tag{13}$$

where higher order terms in these constants are neglected,

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* Dynamicist, Guidance and Control Section, Trajectory Analysis Group.

† For nomenclature see Ref. 5. Equations with R preceding equation number are from Ref. 5.

‡ A similar expression was presented in the comments[8] on Ref. 1.

whereas the corresponding constants for the first-order approximation are

$$\bar{p}_0 = 1 \qquad K = -A/2 \qquad \Phi = \pi/2 \qquad \bar{\tau}_0 = 0$$

as shown in Ref. 5.

Constant Tangential Thrust Acceleration Case (Ref. 6)

Following the same procedures carried out previously, the improved first-order approximation for the present case can be obtained. Equation (2) has its corresponding form as§

$$1/\bar{p} = [(1/\bar{p}_0^4) - 4\epsilon\theta]^{1/4} \tag{14}$$

where ϵ is the constant thrust acceleration in terms of g_0 as defined in Ref. 6. Equations (3–9) for the present case have exactly the same expressions except $1/(A - \tau)$ is replaced by ϵ. The first-order approximation between θ and t has the form

$$\omega(\bar{t} - \bar{t}_0) = (1/\epsilon)\{(1/\bar{p}_0^4) - [(1/\bar{p}_0^4) - 4\epsilon\theta]^{1/4}\} \tag{15}$$

where the boundary condition is $\theta = 0$, $\bar{t} = \bar{t}_0$. Consequently,

$$t = \bar{t} + (6\epsilon\bar{h}^6/K_1\omega)\cos(\theta + \Phi_1) \tag{16}$$

For the case of a satellite initially orbiting in a circular orbit, the integration constants are

$$\bar{p}_0 = 1 - 6\epsilon^2 \qquad K_1 = -(1/2\epsilon)(1 - \tfrac{9}{2}\epsilon^2)$$
$$\Phi_1 = (\pi/2) - \tfrac{3}{2}\epsilon \qquad \bar{t}_0 = 18\epsilon^3/\omega \tag{17}$$

where higher order terms in these constants are neglected, while the corresponding constants for the first-order approximation are

$$\bar{p}_0 = 1 \qquad K = -1/2\epsilon$$
$$\Phi = \pi/2 \qquad \bar{t}_0 = 0$$

Discussion

Geometrically, the trajectory of a satellite subjected to a low tangential thrust is a continuous spiral curve. However, if the concept of an osculating orbit is introduced, Eq. (9) may be conceived as an osculating ellipse. It can be seen that for the case of constant tangential thrust, the osculating orbital elements are

1) Semilatus rectum (l)

$$l = r_0p^2 = r_0(\bar{p} + \delta p)^2$$
$$\simeq r_0(\bar{p}^2 + 2\bar{p}\,\delta p)$$
$$= r_0\bar{p}^2 - [6\bar{p}^5r_0/K_1(A - \bar{\tau})]\sin(\theta + \Phi_1) \tag{18}$$

2) Eccentricity (e)

$$e = p^2S = (\bar{p} + \delta p)^2(\bar{S} + \delta S)$$
$$\simeq \bar{p}^2\bar{S} + 2\bar{p}\bar{S}\,\delta p + \bar{p}^2\delta S$$
$$= (1/K_1\bar{p}) + [2\bar{p}^4/(A - \bar{\tau})][1 - (3/K_1^2\bar{p}^2)] \times$$
$$\sin(\theta + \Phi_1) - [3\bar{p}^3/2K_1(A - \bar{\tau})]\sin 2(\theta + \Phi_1) \tag{19}$$

3) Argument of perigee (Ω)

$$\Omega = -\psi$$
$$= -\Phi_1 - [2K_1\bar{p}^5/(A - \bar{\tau})][\cos(9 + \Phi_1) -$$
$$(3/4K_1\bar{p})\cos 2(\theta + \Phi_1)] \tag{20}$$

In the previous three equations, the first term on the right-hand side depicts the secular portion of the corresponding osculating orbital element, and the remainder terms show their oscillatory characteristics. For the case of constant tangential thrust acceleration, the expression, $1/(A - \bar{\tau})$, in Eqs. (13–20) is replaced by ϵ.

§ In Ref. 6, p is depicted by h.

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Optimum Design of a Vibrating Bar with Specified Minimum Cross Section

J. E. Taylor*
University of California at Los Angeles, Los Angeles, Calif.

Introduction

In a recent study on the optimal design of columns,[1] a method was introduced in which the formulation of the design problem relates to the potential energy of the structure. The optimum column is defined in the study as that design among all columns of given style and equal total volume of material for which the Euler buckling load is a maximum.† Although, this design problem had already received thorough treatment in the literature,[2] the cited energy formulation was presented as a convenient, concise alternate expression of the problem.

A similar technique was applied in a subsequent Note[3] to optimum design problems for vibrating structures. The object of optimization in such problems is to establish, from among all designs of given style and specified total mass, the one for which the lowest natural mode frequency is a maximum.† Problems of this kind are treated in Refs. 4 and 5. In Ref. 3, the design problem is formulated through the statement of a functional which is close to the action integral for the vibrating structure. This expression in terms of energies of the system proved to be convenient for the development of a proof of optimality. The formulation was demonstrated for the design of bars which undergo axial or lateral vibration.

The purpose of this Note is to extend the analytical expression of optimal design problems to accommodate a require-

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* Assistant Professor of Engineering. Member AIAA.

† This design is at the same time the minimum volume (or mass) structure for specified lowest eigenvalue.